EXACT SOLUTIONS WITH SINGULARITIES TO IDEAL HYDRODYNAMICS OF INELASTIC GASES

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Abstract. We construct a large family of exact solutions to the hyperbolic system of 3 equations of ideal granular hydrodynamics in several dimensions for arbitrary adiabatic index γ. In dependence of initial conditions these solutions can keep smoothness for all times or develop singularity. In particular, in the 2D case the singularity can be formed either in a point or along a line. For $\gamma = -1$ the problem is reduced to the system of two equations, related to a special case of the Chaplygin gas. In the 1D case this system can be written in the Riemann invariant and can be treated in a standard way. The solution to the Riemann problem in this case demonstrate an unusual and complicated behavior.

1. Introduction

The motion of the dilute gas where the characteristic hydrodynamic length scale of the flow is sufficiently large and the viscous and heat conduction terms can be neglected is governed by the systems of equations of ideal granular hydrodynamics [3].

This system is given in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, and has the following form:

$$\partial_t \rho + \operatorname{div}_x(\rho v) = 0, \tag{1}$$

$$\partial_t(\rho v) + \operatorname{Div}_x(\rho v \otimes v) = -\nabla_x p, \tag{2}$$

$$\partial_t T + (v, \nabla_x T) + (\gamma - 1) T \operatorname{div}_x v = -\Lambda \rho T^{3/2}, \tag{3}$$

where ρ is the gas density, $v = (v_1, ..., v_n)$ is the velocity, T is the temperature, $p = R\rho T$ is the pressure (the constant R is a adiabatic invariant, for the sake of simplicity we set $R = 1$), and γ is the adiabatic index, $\Lambda = const > 0$. We denote Div_x and div_x the divergence of tensor and vector with respect to the space variables. The only difference between equations (1)–(3) and the standard ideal gas dynamic equations (where the elastic colliding of particles is supposed) is the presence of the inelastic energy loss term $-\Lambda \rho T^{3/2}$ in (3).

The granular gases are now popular subject of experimental, numerical and theoretical investigation (e.g. [3], [7], [8] and references therein). The Navier-Stokes granular hydrodynamics is the natural language for a theoretical description of granular macroscopic flows. A characteristic feature of time-dependent solutions

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of the continuum equations is a formation of finite-time singularities: the density blowup signals the formation of close-packed clusters.

System (1) – (3) can be written in a hyperbolic symmetric form in variables ρ, $v, K = p\rho^{-\gamma}$ and the Cauchy problem

$$(\rho, v, T\rho^{1-\gamma})\big|_{t=0} = (\rho_0, v_0, T_0\rho_0^{1-\gamma})$$

is locally solvable in the class of smooth functions.

System (1) – (3) has no constant solution except the trivial one ($p \equiv 0$). Another trivial solution is $v = p = T \equiv 0$, $\rho(t, x) = \rho_0(x)$. At the same time there exists a solution

$$\rho, v, p = \text{const}, \quad T = T(t) = (\frac{\Lambda\rho_0 t}{2} + T(0)^{-1/2})^{-2}, \tag{4}$$

where $T(0)$ is the initial value of temperature (the Haff's law). This solution is called the homogeneous cooling state.

Here we are going to construct new exact solutions to the ideal granular hydrodynamics with a concentration property and to compare them with the known family of solution obtained earlier in [8].

2. Family of exact solutions in 1D[8]

The authors employ Lagrangian coordinates and derive a broad family of exact non-stationary non-self-similar solutions. These solutions exhibit a singularity, where the density blowups in a finite time when starting from smooth initial conditions. Moreover, the velocity gradient also blowups while the velocity itself and develop a cusp discontinuity (rather then a shock) at the point of singularity.

System (1) - (3) in the Lagrangian coordinates takes the form

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial v}{\partial m}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial m}, \qquad \frac{\partial p}{\partial t} = -\gamma p\rho\frac{\partial v}{\partial m} - \Lambda p^{3/2}\rho^{1/2},$$

and under certain assumptions can be reduced to

$$\frac{\partial^2 p}{\partial m^2} = -\mu^2 p, \qquad \mu = \frac{\Lambda}{\gamma\sqrt{2}}.$$

Here $m(x,t) = \int\limits_0^x \rho(\xi, t)\, d\xi$ is the Lagrangian mass coordinate. The solution is the following:

$$p = 2A\cos(\mu m), \quad A = \text{const}, \quad \rho(m,t) = \frac{\rho(m,0)}{(1-\mu t\sqrt{A\rho(m,0)\cos\mu m})^2}. \tag{5}$$

The rate of concentration at the maximum point of density as $t \to t_*$ is $\rho(0,t) \sim \text{const}(t_* - t)^{-2}$, the behavior of solution at different moments of time and formation of the singularity is presented at Fig.1.

3. Solutions with a constraint

Let us introduce a new dependent variable $z(t, x)$ as follows: $z = \rho - \phi(t)T^{-\frac{1}{2}}$, $\phi(t)$ is an arbitrary differentiable function. Thus, in the variables z, T, v the system (1)–(3) takes the form

$$\partial_t z + \operatorname{div}_x(zv) - \frac{\Lambda}{2}\,\phi(t)\,z + (\gamma+1)\,\phi(t)\,T^{-\frac{1}{2}}\operatorname{div}_x v + (2\phi'(t) + \Lambda\phi^2(t))\,T^{-\frac{1}{2}} = 0, \tag{6}$$

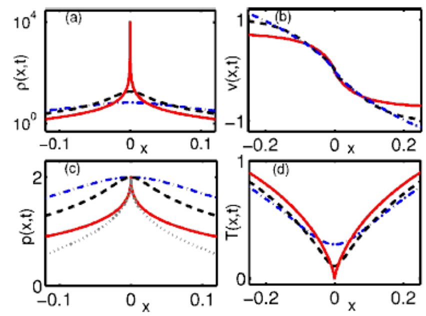

FIGURE 1. Finite mass solution [8]

$$\partial_t v + (v, \nabla_x)\, v = -\frac{1}{z + \phi(t) T^{-\frac{1}{2}}}\, \nabla_x (zT + \phi(t) T^{\frac{1}{2}}), \tag{7}$$

$$\partial_t T + (v, \nabla_x T) + (\gamma - 1)\, T \,\mathrm{div}_x\, v = -\Lambda z\, T^{3/2} - \Lambda\, \phi(t)\, T. \tag{8}$$

We consider a particular class of solutions characterized by property $z = 0$. There are two possibilities:

- $\gamma = -1$ and $\phi(t)$ is a solution to ODE that can be immediately solved:

$$\phi'(t) = -\frac{\Lambda}{2}\phi^2(t), \quad \phi(t) = \left(\frac{\Lambda}{2} t + \rho_0^{-1}(x) T_0^{-\frac{1}{2}}(x)\right)^{-1}, \tag{9}$$

- $v(t, x) = \alpha(t)x + \beta(t)$ and $\phi(t)$ is a solution to ODE

$$\phi'(t) = -\frac{\gamma + 1}{2}\, \phi(t)\, \mathrm{tr}\, \alpha(t) - \frac{\Lambda}{2}\phi^2(t). \tag{10}$$

The first possibility is the case of Chaplygin-like [4] gas, where the state equation is chosen as

$$p = p_0 - \rho^{-1}, \quad p_0 = \mathit{const} > 0. \tag{11}$$

The system (1)–(3),(11) with the constraint $z = 0$ can be reduced to a couple of equations

$$\partial_t v + (v, \nabla_x)\, v = T^{\frac{1}{2}}\, \nabla_x (T^{\frac{1}{2}}), \tag{12}$$

$$\partial_t T^{\frac{1}{2}} + (v,\, \nabla_x T^{\frac{1}{2}}) - T^{\frac{1}{2}} \mathrm{div}_x\, v = -\frac{\Lambda}{2}\, \phi(t)\, T^{\frac{1}{2}}, \tag{13}$$

where $\phi(t)$ is given by (9).

In the 1D case this system as any system of two equations can be written in the Riemann invariants, this allows to apply the technique usual for gas dynamics, we will do this in Sec.3.2.

In the second case, for an arbitrary γ the equation (6) can be satisfied only for $v(t, x) = \alpha(t)x + \beta(t)$, where $\phi(t)$ solves (10). This case will be considered in Sec.3.1.

3.1. **Solutions with uniform deformation, arbitrary γ.** It is known that for usual gas dynamics equations the solutions with linear profile of velocity $v(t,x) = \alpha(t)x + \beta(t)$, where $\alpha(t)$ is a matrix $n \times n$ and $\beta(t)$ is an n - vector, x is a radius-vector of point, constitute a very important class of solutions [10]. For the system of granular hydrodynamics these solutions give a possibility to construct a singularity arising from initial data.

First of all from (12, (13) we get that in this case T has to solve the system

$$(\partial_t \alpha(t) + \alpha^2(t))x + (\partial_t \beta(t) + \alpha(t)\beta(t)) = -\frac{1}{2}\,\nabla_x T, \tag{14}$$

$$\partial_t T + ((\alpha x + \beta), \nabla_x T) + (\gamma - 1) T \,\mathrm{tr}\, \alpha(t) = -\Lambda\, \phi(t)\, T, \tag{15}$$

and the structure of the field of velocity requires a special structure of the field of temperature, namely,

$$T(t,x) = x^T\, A(t)\, x + (B(t), x) + C(t). \tag{16}$$

Thus, we get a system of $\frac{3n^2+5n+4}{2}$ nonlinear differential equations for components of the square matrix $\alpha(t)$, the square symmetric matrix $A(t)$, vectors $\beta(t)$ and $B(t)$, the scalar functions $C(t)$ and $\phi(t)$, namely

$$\alpha'(t) + \alpha^2(t) + A(t) = 0, \qquad \beta'(t) + 2\alpha(t)\beta(t) + \frac{1}{2} B(t) = 0, \tag{17}$$

$$A'(t) + 2A(t)\alpha(t) + (\gamma - 1)\mathrm{tr}\, \alpha(t) A(t) + \Lambda \phi(t) A(t) = 0, \tag{18}$$

$$B'(t) + 2A(t)\beta(t) + B(t)\alpha(t) + (\gamma - 1)\mathrm{tr}\, \alpha(t) B(t) + \Lambda \phi(t) B(t) = 0, \tag{19}$$

$$C'(t) + (B(t), \beta(t)) + (\gamma - 1)\mathrm{tr}\, \alpha(t) C(t) + \Lambda \phi(t) C(t) = 0, \tag{20}$$

and (10). This system can be explicitly (in the simplest cases) or numerically integrated, one can study its qualitative behavior. The component of density can be found as

$$\rho(t,x) = \frac{\phi(t)}{(x^T\, A(t)\, x + (B(t), x) + C(t))^{1/2}}, \tag{21}$$

$\rho(t,x) \sim const * |x - x_0|^{-1}$ in the point x_0 of the singularity formation. Therefore the singularity is integrable for $n > 1$. Nevertheless, the total mass is infinite for this solution, since $\int\limits_{\mathbb{R}^n} \rho\, dx$ diverges as $|x| \to \infty$.

Let us consider the simplest non-rotational case: $A(t) = a(t)\mathbb{I}$, $\alpha(t) = \alpha_1(t)\mathbb{I}$ $B(t) = 0$, $\beta(t) = 0$, where $\mathbb{I}$ is the unit matrix. The system above comes to 4 equations:

$$\phi'(t) + \frac{n}{2}(\gamma + 1)\, \phi(t)\, \alpha_1(t) - \frac{\Lambda}{2}\phi^2(t), \qquad \alpha_1'(t) + \alpha_1^2(t) + a(t) = 0, \tag{22}$$

$$a'(t) + ((2 + n(\gamma - 1))\alpha_1(t) + \Lambda \phi(t)) a(t) = 0, \tag{23}$$

$$C'(t) + (n(\gamma - 1)\alpha_1(t) + \Lambda \phi(t)) C(t) = 0. \tag{24}$$

We are going to find asymptotics of the solution at the point $t = t_* > 0$ of the singularity appearance.

Systems (22)-(24) is a polynomial system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f} : \mathbb{R}^k \to \mathbb{R}^k, \tag{25}$$

and we can study the occurrence of blow-up analyzing the solutions locally around their movable singularities using a set of methods based on the construction of local series. Following [5], [6], we build local series (Ψ-series) of the form:

$$\mathbf{x} = \Psi(\lambda, s, t) = \lambda\tau^s(1 + h(\tau, \ln\tau)),$$

where $\tau = t_* - t$ and $h(\tau; \ln\tau)$ is a power series in its argument which vanishes as $\tau \to 0$. The notation $\lambda\tau^s$ refers to the vector whose i-th component is $\lambda_i\tau^{s_i}$. In order to obtain the leading behavior $\lambda\tau^s$ of the solution around t_* we look for all *negatively quasihomogeneous truncations* $\hat{\mathbf{f}}$ of the vector field $\mathbf{f} = \hat{\mathbf{f}} + \check{\mathbf{f}}$ such that the *dominant behavior* $\mathbf{x} = \lambda\tau^s, \quad \lambda \in \mathbb{C}^k$ is an exact solution of the truncated system $\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x})$ and

$$\check{\mathbf{f}}(\lambda\tau^s) \sim \check{\lambda}\tau^{s+\check{s}-1}, \quad \check{s} \in \mathbb{Q}^k, \quad \check{s}_i > 0,$$

as $\tau \to 0$. Each truncation defines a *dominant balance* (λ, s) and every balance corresponds to the first term $\lambda\tau^s$ in an expansion around movable singularities. For such an expansion to describe a general solution, the Ψ-series must contain $k-1$ arbitrary constants in addition to the arbitrary parameter t_*. The position in the power series where these arbitrary constants appear is given by the *resonances.* They are given by the eigenvalues of the matrix R:

$$R = -D\hat{\mathbf{f}}(\lambda) - diag(s),$$

where $D\hat{\mathbf{f}}(\lambda)$ is the Jacobian matrix evaluated on λ. The resonances are labeled r_i, $i = 1, ..., k$ with $r_1 = -1$. Each balance defines a new set of resonances.

Theorem 3.1. *([6], T.3.8) Consider a real analytic system (25)and assume that it has a balance (λ, s) such that $r_j > 0$ for all $j > k - m + 1$, $1 \le m \le k$ and $\lambda \in \mathbb{R}^n$. Then there exists a m-dimensional manifold $S_0^m \subseteq \mathbb{R}^k$ of initial conditions leading to a finite time blow-up, for all $x_0 \in S_0^m$, i.e. there exists $t_* \in \mathbb{R}_+$ for which $|\mathbf{x}(t; x_0)| \to \infty$ as $t \to t_*$.*

Corollary 3.1. *If $n \ge 2$ and $\gamma > -1 + \frac{2}{n}$, then there exists an open set Ω of initial data $x_0 = (\phi(0), \alpha_1(0), a(0), C(0))$ such that the components $\phi(t)$ and $\alpha_1(t)$ of solution to the system (22)–(24) blow up within a finite time for all $x_0 \in \Omega$.*

Proof. To find main terms of asymptotic at the point of singularity we consider a negatively quasihomogeneous truncation of the system (22)–(24), namely

$$\phi'(t) = -\frac{n}{2}(\gamma + 1)\,\phi(t)\,\alpha_1(t) - \frac{\Lambda}{2}\phi^2(t), \tag{26}$$

$$\alpha_1'(t) = -\alpha_1^2(t), \tag{27}$$

$$a'(t) = -(2 + n(\gamma - 1))\alpha_1(t)a(t) - \Lambda\phi(t)a(t), \tag{28}$$

$$C'(t) = -n(\gamma - 1)\alpha_1(t)C(t) - \Lambda\phi(t)C(t). \tag{29}$$

The solution to the above system is the following:

$$\phi(t) = -\frac{n(\gamma + 1) - 2}{\Lambda}(t_* - t)^{-1}, \quad \alpha_1(t) = -(t_* - t)^{-1}, \tag{30}$$

$$A(t) = A_0(t_* - t)^{2(n-2)}, \quad C(t) = C_0(t_* - t)^{2(n-1)}, \qquad A_0, C_0 = \text{const.},$$

$$s = diag(-1, -1, 2(n-2), 2(n-1)), \quad \lambda = (-1, -\frac{n(\gamma + 1 - 2)}{\Lambda}\, A_0,\, C_0).$$

The resonances, computed for this balance are $(n(\gamma+1)-2,-1,0,0)$. Theorem 3.1 result that there exists a manyfold $S_0^2 \in \mathbb{R}^n$ such that for $\phi(0), \alpha_1(0)$ the respective solution to (22) blows up and has (30) as a main term of asymptotics. Other components of solution to (22)– (24) can be found from linear with respect to $a(t)$ and $C(t)$ equations (28) and (29) for any initial data (it makes sense to consider $a(0)>0, C(0)>0$). $\square$

Remark 3.1. The rate of growth of the maximum of the density as $t \to t_*$ is $\rho(t,0) \sim \mathrm{const}(t-t_*)^{-n}$.

Remark 3.2. If $t_* < 0$, then an analogous consideration shows that there exists an open set of initial data such that the solution to system (22)–(24)remains bounded for all $t > 0$.

Theorem 3.2. *It $n=1$, then for any $t_*>0$ there exists a family of exact solutions to system (22)– (24) depending on parameters (α_0, C_0), blowing up as $t \to t_*$. This family is physically reasonable for $\alpha_0 \in (-1, -\frac{2}{\gamma+1})$, $\quad C_0 > 0, \gamma > 1$. For these solutions the the maximum of density has the asymptotics $\rho(t,0) \sim \mathrm{const}(t-t_*)^{\alpha_0}$ as $t \to t_*$.*

Proof. It can be readily checked that the balance $s = (-1,-1,-2,-2(\alpha_0+1))$, $\lambda = (-(2+(\gamma+1)\alpha_0)/\Lambda, \alpha_0, -\alpha_0(\alpha_0+1), C_0)$ gives an exact solution. The restriction on the parameters α_0 and C_0 follows from the positivity of the expression under the square root in (21) and the positivity of $\phi(0)$. $\square$

Remark 3.3. The maximum of density as $t \to t_*$ grows slower than for the solution obtained in [8].

Fig.2a presents the results of numerical computations in 2D based on system (17) –(29). The initial density has the form (21). Fig.2b shows the density near the blow-up time for $\alpha_{11} = \alpha_{22} < 0$, $\alpha_{12} = \alpha_{21} = 0$, with a concentration in a point. Fig.2c shows the density near the blow-up time for $\alpha_{11} < 0$, $\alpha_{22} = \alpha_{12} = \alpha_{21} = 0$ with a concentration along a line. The computations demonstrate a complicated behavior of solution. In particular, a vorticity can prevent the singularity formation.

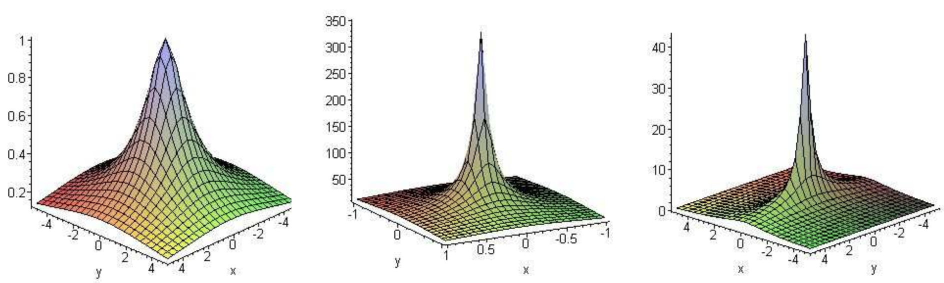

FIGURE 2. a,b,c

3.2. **Chaplygin gas,** $n = 1$**.** The model of gas dynamics with the pressure given by (11) is known as the Chaplygin gas. The Chaplygin gas is now considered as a possible model for dark matter-energy [1]. This system can also be seen as the one-dimensional version of the Born-Infeld system, a non linear modification of the Maxwell equations, designed by Born and Infeld in 1934 to solve the electrostatic divergence generated by point particles in classical electrodynamics.The Chaplygin system is known to be hyperbolic, linearly degenerate, weakly stable [12]. Recently this system attracted a lot of attention, e.g [2], [11].

The system (12), (13) can be reduced to

$$\partial_t \rho + \mathrm{div}_x(\rho v) = 0, \qquad \partial_t(\rho v) + \mathrm{Div}_x(\rho v \otimes v - \frac{\phi^2(t)}{\rho}) = 0, \tag{31}$$

recall that $\rho = \phi(t) T^{-\frac{1}{2}}$. It is similar to the Chaplygin gas system, the only difference is in the known multiplier $\phi(t)$, for the Chaplygin gas $\phi = const$. For $n = 1$ (31) can be written in the Riemann invariants as

$$\partial_t s + r \partial_x s = \frac{\Lambda \phi(t)}{4}(r - s), \qquad \partial_t r + s \partial_x r = \frac{\Lambda \phi(t)}{4}(s - r), \tag{32}$$

where $s = v - T^{\frac{1}{2}}$, $r = v + T^{\frac{1}{2}}$, $\phi(t)$ is given by (9). This system is linear degenerate, therefore provided the solution is bounded there is no possibility for the gradient catastrophe.

Theorem 3.3. *The solution to the Riemann problem for (31) ((32)) in 1D with data*

$$(v, T) = \begin{cases} (v_L, T_L), & x < 0, \\ (v_R, T_R), & x > 0, \end{cases} \tag{33}$$

in the case

$$v_L > v_R \tag{34}$$

contains a δ-singularity in the component of density. If

$$v_L \ge v_R + T_L^{\frac{1}{2}} + T_R^{\frac{1}{2}}, \tag{35}$$

then the δ-singularity formation begins from the initial moment of time.

Proof. Since the system is linear degenerate, the jumps are contact discontinuities and move along characteristics. The solution is based on the cooling state (4). If

$$v_L < v_R + T_L^{\frac{1}{2}} + T_R^{\frac{1}{2}}, \tag{36}$$

then the solution is

$$(v, T) = \begin{cases} (v_L, T_L(t)), & x < x_-(t), \\ (v_M(t), T_M(t)), & x_-(t) < x < x_+(t), \\ (v_R, T_L(t)), & x > x_+(t), \end{cases} \tag{37}$$

with $c = \phi^{-1}(0)$, $T_L(t) = \frac{c^2 T_L}{(\frac{\Lambda}{2}t + c)^2}$, $T_R(t) = \frac{c^2 T_R}{(\frac{\Lambda}{2}t + c)^2}$,

$$x_-(t) = v_L t - \frac{2c T_L^{\frac{1}{2}}}{\Lambda} \ln(\frac{\Lambda}{2c} t + 1) \quad x_+(t) = v_R t + \frac{2c T_R^{\frac{1}{2}}}{\Lambda} \ln(\frac{\Lambda}{2c} t + 1),$$

$$v_M(t) = \frac{v_L + v_R + c(T_R^{\frac{1}{2}} - T_L^{\frac{1}{2}})(\frac{\Lambda}{2}t + c)^{-1}}{2},$$

$$T_M^{\frac{1}{2}}(t) = \frac{v_R - v_L + c(T_R^{\frac{1}{2}} + T_L^{\frac{1}{2}})(\frac{\Lambda}{2}t + c)^{-1}}{2}.$$

If $v_L \le v_R$, then $x_-(t) < x_+(t)$ for all $t > 0$ and the solution to the Riemann problem is given by (37). If $v_L < v_R$, then there exists a moment $t_* > 0$ such that $x_-(t_*) = x_+(t_*)$. Moreover, in the moment $t_{**} > 0$ the component T_M vanishes and $t_{**} < t_*$. Thus, we have to construct a new solution starting from t_{**}.

We are going to introduce a δ-singularity in the density concentrated on the jump analogously to [2]. To find a δ-type singularity solution we have to use the system in its conservative form (31). Let us denote $x_*(t)$ the position of the singularity and look for a solution in the form:

$$\rho(t,x) = \rho_- + [\rho]H(x - x_*(t)) + \theta(t)\delta(x - x_*(t)), \tag{38}$$

$$\rho(t,x)v(t,x) = \rho v_- + [\rho v]H(x - x_*(t)) + \psi(t)\delta(x - x_*(t)), \tag{39}$$

$$\rho(t,x)v^2(t,x) = \rho v_-^2 - +[\rho v^2]H(x - x_*(t)) + \Psi(t)\delta(x - x_*(t)), \tag{40}$$

$$\tau(t,x) = \tau_- + [\tau]H(x - x_*(t)), \quad \tau = \rho^{-1}, \tag{41}$$

$[f] = f_+ - f_-$, f_+ and f_- are the limits of an arbitrary function from the right and from the left side of $x_*(t)$, respectively, H is the Heaviside function. From (31) we get

$$x_*(t) = \frac{[\rho v]t_1 + \theta(t)}{[\rho]} + x_0, \tag{42}$$

$$\theta(t) = \sqrt{\left([\rho v]^2 - [\rho][\rho v^2]\right)t_1^2 + \frac{4}{\Lambda}[\rho][\tau]\left(\phi(t_0)t_1 - \frac{2}{\Lambda}\ln\left(\frac{\Lambda\phi(t_0)}{2}t_1 + 1\right)\right)}, \tag{43}$$

$t_1 = t - t_0$, t_0 and x_0 are the moment and the coordinate of the δ-singularity formation. Expanding the expression under the square root at $t = t_0$ we find the necessary condition for the beginning of the concentration processes:

$$[v]^2 \ge [T^{1/2}(t_0)]^2. \tag{44}$$

If the initial data are such that the inequality opposite to (36) holds, i.e. $v_L \ge v_R + T_L^{1/2}(0) + T_R^{1/2}(0)$, then $v_L > v_R$ and

$$(v_L - v_R)^2 \ge (T_L^{1/2}(0) + T_R^{1/2}(0))^2 \ge (T_R^{1/2}(0) - T_L^{1/2}(0))^2,$$

such that the condition (44) is satisfied. Since in this case the solution consisting of two contact discontinuities is impossible, the only reasonable solution is given by (38)– (41), (42).

If $u_L > u_R$, we can define again the solution of form (38)– (41), (42). In principle, initially the condition (44) may fail and this solution can exist beginning from some $\hat{t}$, $0 < \hat{t} \le t_{**}$, nevertheless, this moment $\hat{t}$ always exists. Further, at least for $t > t_*$ this solution this solution is stable. Let us extend this solution back $t_{**} < t < t_*$. This means the assumption that the segment $[x_-(t), x_+(t)]$ shrinks into the point $x_*(t)$ at the moment t_{**}. It can be shown that the velocity of the singular front $\dot{x}_*(t) \to \frac{v_L + v_R}{2}$ as $t \to \infty$. $\square$

Remark 3.4. The question on uniqueness of the solution after the "shrinking" is open.

Remark 3.5. In [9] for any spatial dimensions a simple family of solutions to the system (1) – (3) having a singularity in the density whereas other components are continuous is constructed. Moreover, a family of self-similar solutions in 1D was found.

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